APPENDIX E

ADIABATIC COMPRESSIBLE FLOW WITH FRICTION, USING MACH NUMBER AS A PARAMETER

This appendix gives derivations for application equations presented in Chapter 4.

Street et al. [1] and Shapiro [2] give the following relation for a constant-area duct flowing a gas *with sonic velocity at the exit*:

$$f_{\text{ave}}\frac{L_{\max}}{D} = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma}\ln\left[\frac{(\gamma+1)M^2}{2\left(1+\frac{\gamma-1}{2}M^2\right)}\right], \qquad \text{(4.16, repeated)}$$

where

f_{ave} = average Darcy friction factor along the duct,

$L_{\max}$ = maximum attainable duct length with M at the inlet, ft (or m),

D = duct diameter, ft (or m),

γ = ratio of specific heats of flowing gas, and

M = Mach number of the gas flow at the duct inlet.

In the development of this equation, f is assumed to be a constant, and f_{ave} is taken as a reasonable value for f. In actuality, of course, since fluid temperature changes continuously along the duct, the fluid viscosity also changes, and then so does Reynolds Number—resulting in a varying friction factor. But it turns out that the variation is modest enough to be handled by using the average friction factor.

E.1 SOLUTION WHEN *STATIC* PRESSURE AND *STATIC* TEMPERATURE ARE KNOWN

Equation 4.16 may be used to find the $L_{\max}$ of the duct if the essential duct data are available: flow rate, inlet static pressure, inlet static temperature, duct diameter, friction factor, and gas ratio of specific heats, molecular weight, and compressibility factor. The Mach number of a gas flowing in a duct (assuming a flat velocity profile) is:

$$M = \frac{u}{\mathrm{A}} \approx \frac{V}{\mathrm{A}}. \qquad \text{(1.4, repeated)}$$

The equation for the acoustic velocity A is:

$$\mathrm{A} = \sqrt{\frac{\gamma P}{\rho_m}}\text{(mass units)} \quad \text{or} \quad \mathrm{A} = \sqrt{\frac{\gamma P g}{\rho_w}}\text{(weight units)}.$$

Utilizing the equation of state, Equations 1.6 in Chapter 1, the acoustic velocity may be expressed as:

$$\mathrm{A} = \sqrt{\frac{\gamma P}{\rho_m}} = \sqrt{\frac{\gamma P}{P/zRT}} = \sqrt{\gamma zRT} = \sqrt{\gamma z\frac{\bar{R}}{m}T} \qquad \text{(E.1a)}$$
$$\left(\bar{R} = 8314.34\ \mathrm{J/mol_{Kg}{}^\circ K}\right),$$

or

$$\mathrm{A} = \sqrt{\frac{\gamma P}{\rho_w}} = \sqrt{\frac{\gamma P g}{P/zRT}} = \sqrt{\gamma gzRT} = \sqrt{\gamma gz\frac{\bar{R}}{m}T} \qquad \text{(E.1b)}$$
$$\left(\bar{R} = 1545.31\ \text{ft-lb/mol}_{\text{lb}}{}^\circ\text{R}\right).$$

Pipe Flow: A Practical and Comprehensive Guide, First Edition. Donald C. Rennels and Hobart M. Hudson.

(In these two equations, m represents molecular weight, not mass.) The compressibility factor z may be evaluated using one of the formulas found in Appendix D. Utilizing Equations E.1a and E.1b, and $\dot{m} = AV\rho_m$ and $\dot{w} = AV\rho_w$ from Chapter 2, we may write:

$$M = \frac{\dot{m}/A\rho_m}{\sqrt{\gamma z \frac{\bar{R}}{m} T}} = \frac{\dot{m}}{A\frac{Pm}{z\bar{R}T}\sqrt{\gamma z \frac{\bar{R}}{m} T}} = \frac{\dot{m}}{AP\sqrt{\gamma \frac{m}{z\bar{R}T}}} = \frac{\dot{m}}{AP}\sqrt{\frac{z\bar{R}T}{\gamma m}} \tag{E.2a}$$

or

$$M = \frac{\dot{w}/A\rho_w}{\sqrt{\gamma g z \frac{\bar{R}}{m} T}} = \frac{\dot{w}}{A\frac{Pm}{z\bar{R}T}\sqrt{\gamma g z \frac{\bar{R}}{m} T}} = \frac{\dot{w}}{AP\sqrt{\gamma g \frac{m}{z\bar{R}T}}} = \frac{\dot{w}}{AP}\sqrt{\frac{z\bar{R}T}{\gamma g m}}. \tag{E.2b}$$

Using this Mach number, evaluated at the duct inlet, L_{max} becomes immediately available from Equation 4.16.

Equation 4.16 may not be violated.* The length of the duct may not exceed L_{max} with sonic velocity ($M = 1$) occurring at the exit. However, if the length of the duct is less than L_{max} as given by Equation 4.16, then the exit Mach number will be less than unity. This is the most frequently encountered case.

Consider a gas receiver discharging through a round duct of known length L_{line} to a lower pressure region and suppose that the pressure conditions are such that the discharging gas exits from the duct at subsonic velocity (see Fig. E.1). Assume that friction factor f and diameter D are constant. If we know the flowing conditions at one end—either end—of the duct (flow rate, duct diameter, pressure, and temperature), we may find the Mach number M there and then use Equation 4.16 to find the $(fL/D)_{limit}$ at that end of the duct. By Equation 8.1, this can be called K_{limit} at that end. (Remember that because f and D are constant, K in this context is simply length with a constant coefficient.) Note that since the flow exits from the duct subsonically, this K_{limit} includes a virtual length of duct at which the flow would attain sonic velocity (provided that the pressure at the virtual outlet was low enough). Now, because f/D is constant, K is proportional to L so that we can write:

$$(K_1)_{limit} = K_{line} + (K_2)_{limit}. \tag{E.3}$$

Knowing the line resistance coefficient K_{line} and limit resistance coefficient $(K)_{limit}$ at one end of the duct enables us to find the limit resistance coefficient at the other end of the duct. Then, since $(K)_{limit}$ is associated with M at that end by Equation 4.16, we may find M at that end by solving the equation.

Because Equation 4.16 cannot be solved for M explicitly, it must be solved by trial and error. The Newton–Raphson method is a convenient method for the solution. In order to implement it, we need to rearrange the equation so that we have an *expression that equals zero*. We can do this by subtracting $(K_1)_{limit}$ from both sides of Equation E.3:

$$0 = (K_2)_{limit} - [(K_1)_{limit} - K_{line}]. \qquad \text{(E.3, rearranged)}$$

Now $(K_1)_{limit} - K_{line} = (K_2)_{limit}$, and while we know the values of $(K_1)_{limit}$, K_{line}, and $(K_2)_{limit}$, we do not know the value of the Mach number yielding $(K_2)_{limit}$, and we are interested in knowing this value so that we may find the flowing conditions at the actual duct outlet. Let us call K_i the guessed value of $(K_2)_{limit}$ and write:

$$f(M) = K_i - (K_2)_{limit} = 0. \tag{E.4}$$

This expression is supposed to equal zero, and it will be if we evaluate K_i using the right Mach number. If we guess a Mach number and evaluate K_i by Equation 4.16, the result is not likely to equal K_{limit} and $f(M)$ is not likely to equal zero. This is shown graphically in Figure E.2.

If we extrapolate down the function's tangent, it is clear that at the intersection with $f(M) = 0$, we will find a much better guess for M. To do this requires the derivative of K with respect to M:

$$\frac{dK_{limit}}{dM} = -\frac{2}{\gamma M^3} + \frac{\gamma+1}{\gamma M}\left[\frac{1}{1+\frac{\gamma-1}{2}M^2}\right]. \tag{E.5}$$

* This is not to say that supersonic flow cannot occur in a constant area duct; it can, but the flow must be introduced to the duct in a supersonic condition.

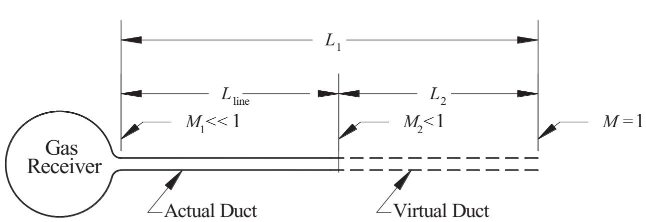

FIGURE E.1. Subsonic constant-area gas flow duct (Fig. 4.5, repeated).

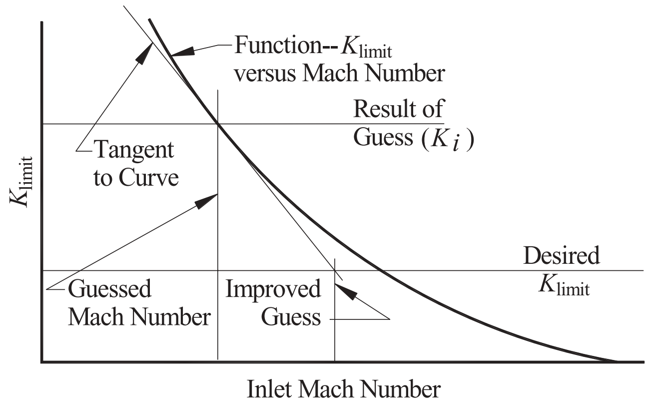

FIGURE E.2. Mach number solution by the Newton–Raphson method.

Now a better approximation of M may be found with the extrapolation formula:

$$M_{i+1} = M_i - \frac{K_i - K_{\text{limit}}}{(dK_{\text{limit}}/dM)}, \tag{E.6}$$

where M_{i+1} is the improved approximation and M_i is the earlier or guessed value. As the natural logarithm term in Equation 4.16 is much smaller than the preceding term, use the approximation:

$$K_{\text{limit}} \approx \frac{1-M^2}{\gamma M^2}$$

or

$$M \approx \sqrt{\frac{1}{1+\gamma K_{\text{limit}}}} \tag{E.7}$$

for the first guess of M. This guess for M may be entered in Equation 4.16 to find K_i, the estimated limit on K based on M. Enter it also in Equation E.5 to get dK_{limit}/dM. Then enter all three variables in Equation E.6 to obtain an improved estimate of M. Repeat the process to get K_i and dK_{limit}/dM at the new, better estimate of M, and then a much improved estimate of M.

After several iterations, the second term in the iteration formula will become quite small and the successive approximations of M will become more nearly alike. When the corrections become as small as desired (say, one part in a million), the iterations may be halted and the Mach number considered solved.

Once the unknown Mach number is found, the accompanying pressure and temperature may be found. The static pressure, in terms of the local Mach number and the static pressure P_* at the location where Mach number is unity (that is, where velocity is sonic) is given by:

$$\frac{P}{P_*} = \frac{1}{M}\sqrt{\frac{\gamma+1}{2\left(1+\frac{\gamma-1}{2}M^2\right)}}. \tag{E.8}$$

Taking the ratio of the expression evaluated for $M = M_1$ to that for $M = M_2$ yields:

$$\frac{P_1}{P_2} = \frac{M_2}{M_1}\sqrt{\frac{1+M_2^2(\gamma-1)/2}{1+M_1^2(\gamma-1)/2}}, \tag{E.9}$$

from which the desired pressure is easily found. The static temperature is available similarly from:

$$\frac{T}{T_*} = \frac{\gamma+1}{2\left(1+\frac{\gamma-1}{2}M^2\right)}. \tag{E.10}$$

The ratio of the inlet and outlet static temperatures is thus:

$$\frac{T_1}{T_2} = \frac{1+M_2^2(\gamma-1)/2}{1+M_1^2(\gamma-1)/2}, \tag{E.11}$$

from which the desired temperature is easily found.

The foregoing relationships are useful if the static pressure and static temperature at one end of the duct are known. If one or the other of the static values is not known, but the corresponding total value is known (and this is often, if not usually, the case) these equations may still be solved, but account must be made for the divergence between total and static values. For instance, if a gas in a pressurized vessel is allowed to escape to atmosphere through a duct and it attains sonic velocity at the end of the conduit, the static pressure at the outlet end of the duct may be as low as half its total pressure and static temperature may be as low as 80% of its total temperature.

There are three cases in which the required static values are not all known: (1) static pressure and total temperature are known; (2) total pressure and total temperature are known; and (3) total pressure and static temperature are known. These will be considered in order. We must make use of the following relationships:

$$T = \frac{T_t}{1+M^2(\gamma-1)/2}, \tag{E.12}$$

$$P = \frac{P_t}{\left[1+M^2(\gamma-1)/2\right]^{\gamma/(\gamma-1)}}, \tag{E.13}$$

where T, P, T_t, P_t, and M are local values (i.e., all at the same location).

In order to simplify the equations, let us recast the equation for Mach number (Eq. E.2a or E.2b) in the following form:

$$M = B\sqrt{T}/P, \tag{E.14}$$

where

$$B = \frac{\dot{m}}{A}\sqrt{\frac{z\bar{R}}{\gamma m}} \quad \left(\bar{R} = 8314.34\ \mathrm{J/mol_{Kg}\,°K}\right) \tag{E.15a}$$

or

$$B = \frac{\dot{w}}{A}\sqrt{\frac{z\bar{R}}{\gamma g m}} \quad \left(\bar{R} = 1545.31\ \text{ft-lb/mol}_{\text{lb}}\,°\text{R}\right). \tag{E.15b}$$

E.2 SOLUTION WHEN *STATIC* PRESSURE AND *TOTAL* TEMPERATURE ARE KNOWN

Now, if *static pressure* and *total temperature* are known, substitute the expression for static temperature T (Eq. E.12), in terms of total temperature T_t, in place of T; then:

$$M = \frac{B}{P}\sqrt{\frac{T_t}{1+M^2(\gamma-1)/2}}. \tag{E.16}$$

This equation is a quadratic in M^2 whose solution is:

$$M^2 = \frac{\sqrt{1+2(\gamma-1)(B\sqrt{T_t}/P)^2}-1}{\gamma-1}. \tag{E.17}$$

Note the similarity of the expression $B\sqrt{T_t}/P$ in Equation E.17 to that for Mach number M in Equation E.14. They are identical except that the one above contains T_t while Equation E.14 contains simply T. Let us therefore call the expression (and similar expressions utilizing the available temperature and pressure, whether they be static or total) "core Mach number," M_{core}, because of its similarity to the simple expression for Mach number based on static values, and because it is the "core" of the expression for Mach number when other than static values are utilized. Then, for the *static pressure* and *total temperature* case, we may write:

$$M^2 = \frac{\sqrt{1+2(\gamma-1)M_{\text{core}}^2}-1}{\gamma-1}. \tag{E.18}$$

This M^2 may now be substituted into Equation 4.16 to find the $f_{\text{ave}}L_{\text{max}}/D$ or K_{limit}, and from thence to find the Mach number at the other end of the duct and the accompanying pressure and temperature.

E.3 SOLUTION WHEN *TOTAL* PRESSURE AND *TOTAL* TEMPERATURE ARE KNOWN

If *total pressure* and *total temperature* are known at one end of the duct, the expressions for static pressure in terms of total pressure and static temperature in terms of total temperature may be substituted into Equation E.14 to obtain the equation for M. But in order to simplify the algebra, let us simplify the equations for T_t and P_t (Eqs. E.12 and E.13) by substituting the parameter X for the expression $1 + M^2(\gamma - 1)/2$:

$$T = \frac{T_t}{1+M^2(\gamma-1)/2} = \frac{T_t}{X}, \tag{E.19}$$

$$P = \frac{P_t}{\left[1+M^2(\gamma-1)/2\right]^{\gamma/(\gamma-1)}} = \frac{P_t}{X^{\gamma/(\gamma-1)}}. \tag{E.20}$$

Now Equation E.14 may be written as:

$$M = B\frac{\sqrt{T}}{P} = B\sqrt{\frac{T_t}{X}}\frac{X^{\gamma/(\gamma-1)}}{P_t} = B\frac{\sqrt{T_t}}{P_t}X^{\gamma/(\gamma-1)}X^{-1/2} = M_{core}X^{(\gamma+1)/2(\gamma-1)}. \tag{E.21}$$

Squaring and substituting $1 + M^2(\gamma - 1)/2$ for X yields:

$$M^2 = M_{\text{core}}^2\left[1+M^2(\gamma-1)/2\right]^{(\gamma+1)/(\gamma-1)}. \tag{E.22}$$

Equation E.22 cannot be solved explicitly. Using the Newton–Raphson iterative method, however, it is easily solved. The solution is simpler if we use our parameter X as the variable. In the equation $X = 1 + M^2(\gamma - 1)/2$, solve for M^2:

$$M^2 = \frac{2(X-1)}{\gamma-1}. \tag{E.23}$$

Now substitute these expressions into Equation E.22 and solve for zero:

$$\frac{2(X-1)}{\gamma-1} = M_{\text{core}}^2 X^{(\gamma+1)/(\gamma-1)}, \tag{E.24}$$

$$0 = \frac{\gamma-1}{2}M_{\text{core}}^2 X^{(\gamma+1)/(\gamma-1)} - X + 1. \tag{E.25}$$

In the Newton–Raphson method, we need to set this function equal to $f(X)$ and differentiate in order to find the value of X when the function is equal to zero. The derivative of $f(X)$ is:

$$f'(X) = \frac{\gamma+1}{2}M_{\text{core}}^2 X^{2/(\gamma-1)} - 1. \tag{E.26}$$

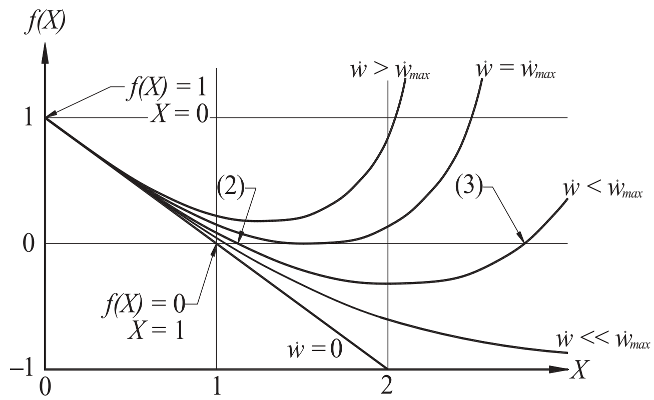

FIGURE E.3. Graph of $f(X)$ versus X.

Using the functions for $f(X)$ and $f'(X)$ defined above, any degree of precision may be obtained by repeated application of:

$$X_{i+1} = X_i - \frac{f(X_i)}{f'(X_i)}, \tag{E.27}$$

where X_i is an estimate and X_{i+1} is a much closer estimate. After several successive iterations when the value of $f(X)$ is sufficiently close to zero, the value of X will be established. Then M may be found from Equation E.23 and Equation 4.16 evaluated for K.

A pitfall in employing this technique lies in assuming the equation has a solution. The graph of $f(X)$ versus X is illustrated in Figure E.3. If the flow rate is 0 then $M_{\text{core}} = 0$ and $f(X)$ crosses the zero axis at $X = 1$. As $\dot{w}$ is increased, the curve moves up and crosses the zero axis in two places, points (2) and (3) in the illustration, so there are actually two solutions—one is subsonic and one is supersonic. Depending on the value of your initial guess for $X_{i=0}$ your solution for M might be either the subsonic one or the supersonic one.

As $\dot{w}$ is increased more, the $f(X)$ curve intersections of the $f(X) = 0$ line become closer together; then, when the crossings coincide the $f(X)$ curve becomes tangent to the zero axis, and $M = 1$, the flow is sonic at the point of interest. At this point $\dot{w}$ is maximized and becomes $\dot{w}_{\max}$. If $\dot{w}$ is increased further, $f(X)$ does not intersect the zero axis and there is no solution. This indicates that for any given total pressure and total temperature condition, flow in a constant area duct cannot exceed a discrete value where Mach number at the outlet becomes unity.

The difficulty described above may be easily avoided by making the following test. At the minimum value of $f(X)$, $f'(X) = 0$:

$$f'(X) = \frac{\gamma+1}{2} M_{\text{core}}^2 X^{2/(\gamma-1)} - 1 = 0. \tag{E.28}$$

Therefore, at $f'(X) = 0$, where $f(X) = f(X)_{\min}$, X is:

$$X = \left(\frac{\gamma+1}{2} M_{\text{core}}^2\right)^{-(\gamma-1)/2}. \tag{E.29}$$

Substituting this value for X into the expression for $f(X)$ (see Eq. E.25) we find that:

$$f(X)_{\min} = \frac{\gamma-1}{2} M_{\text{core}}^2 \left(\frac{\gamma+1}{2} M_{\text{core}}^2\right)^{-(\gamma+1)/2} - \left(\frac{\gamma+1}{2} M_{\text{core}}^2\right)^{-(\gamma-1)/2}. \tag{E.30}$$

- If $f(X)_{\min} < 0$, two solutions exist as at (2) and (3), and since in duct flow we are interested in the subsonic solution, our initial guess for X, that is, $X_{i=0}$, must be less than X at $f(X)_{\min}$ (that is, X from Eq. E.29).
- If $f(X)_{\min} = 0$, this is the limiting condition, and may be treated accordingly.
- If $f(X)_{\min} > 0$, there is no solution, the input conditions are impossible, and the calculation may be halted or redirected, as, for instance, making the pipe diameter larger or reducing the flow rate, depending on what part of your design you are pursuing. If your design has a fixed flow rate, you can increase the pipe size. If your design has a fixed pipe size, you can reduce the flow rate to determine what flow it can handle and from this, you can determine the accompanying pressures and temperatures.

E.4 SOLUTION WHEN *TOTAL* PRESSURE AND *STATIC* TEMPERATURE ARE KNOWN

The equations for solving for M if *total pressure* and *static temperature* are given are similar to those derived above for total pressure and total temperature, and are derived similarly. Mach number is given by Equation E.14,

$$M = B\sqrt{T}/P, \tag{E.14, repeated}$$

where B is defined by Equation E.15a or E.15b. In this case, static temperature is already known, but the known pressure is total pressure, from which static pressure must be determined using Equation E.13, which is

$$P = \frac{P_t}{\left[1 + M^2(\gamma-1)/2\right]^{\gamma/(\gamma-1)}}. \tag{E.13, repeated}$$

Substituting this expression for P in Equation E.14 yields:

$$M = B\sqrt{T}/P = \frac{B\sqrt{T}}{P_t}\left[1+M^2(\gamma-1)/2\right]^{\gamma/(\gamma-1)}.$$

We have previously defined $B\sqrt{T}/P$ as M_{core} without regard as to whether T or P is total or static, so we can write the equation as:

$$M = M_{core}\left[1+M^2(\gamma-1)/2\right]^{\gamma/(\gamma-1)}.$$

Upon squaring and substituting X for $1 + M^2(\gamma - 1)/2$, the equation becomes:

$$M^2 = M_{\text{core}}^2\left[1+M^2(\gamma-1)/2\right]^{2\gamma/(\gamma-1)} = M_{\text{core}}^2 X^{2\gamma/(\gamma-1)}. \tag{E.31}$$

Solving the equation $X = 1 + M^2(\gamma - 1)/2$ for M^2 yielded Equation E.23:

$$M^2 = \frac{2(X-1)}{\gamma-1}, \qquad \text{(E.23, repeated)}$$

which, when substituted in Equation E.31, gives:

$$\frac{2(X-1)}{\gamma-1} = M_{\text{core}}^2 X^{2\gamma/(\gamma-1)}.$$

If we rearrange this and make the rearrangement equal zero we obtain:

$$0 = \frac{\gamma-1}{2}M_{core}^2 X^{2\gamma/(\gamma-1)} - X + 1.$$

If we call the right side of this equation $f(X)$ we get a function that is supposed to equal zero (but it won't equal zero unless we discover the right value for X):

$$f(X) = \frac{\gamma-1}{2}M_{\text{core}}^2 X^{2\gamma/(\gamma-1)} - X + 1. \tag{E.32}$$

In order to find X we need the derivative of Equation E.32, which is

$$f'(X) = \gamma M_{\text{core}}^2 X^{(\gamma+1)/(\gamma-1)} - 1. \tag{E.33}$$

Equations E.32 through E.35 should be applied in the same fashion as Equations E.19 through E.30. Using the functions for $f(X)$ and $f'(X)$ defined above, any degree of precision may be obtained by repeated application of:

$$X_{i+1} = X_i - \frac{f(X_i)}{f'(X_i)}. \qquad \text{(E.27, repeated)}$$

The value of X at $f'(X) = 0$ is:

$$X = \left(\gamma M_{\text{core}}^2\right)^{-(\gamma-1)/(\gamma+1)}. \tag{E.34}$$

The value of $f(X)_{\min}$ is:

$$f(X)_{\min} = \frac{\gamma-1}{2}M_{\text{core}}^2\left(\gamma M_{\text{core}}^2\right)^{-2/(\gamma+1)} - \left(\gamma M_{\text{core}}^2\right)^{-(\gamma-1)/(\gamma+1)}. \tag{E.35}$$

The caveats following those equations are also the same for this case:

- If $f(X)_{\min} < 0$, two solutions exist, and since in duct flow what we are interested in is the subsonic solution, our initial guess for X, that is, $X_{i=0}$, must be less than X at $f(X)_{\min}$ (that is, X from Eq. E.34 for this case). By making the first guess for X (i.e., $X_{i=0}$) less than X at $f(X)_{\min}$, Equation E.27 searches for the solution on the part of the curve where $f'(X)$ is negative, the descending part of the curve. The subsonic solution lies somewhere on the descending part of the curve and the supersonic solution lies on the ascending part of the curve.
- If $f(X)_{\min} = 0$, this is the limiting condition, and may be treated accordingly.
- If $f(X)_{\min} > 0$, there is no solution, the input conditions are impossible, and the calculation may be halted or redirected, as, for instance, making the pipe diameter larger or reducing the flow rate, depending on what part of your design you are pursuing. If your design has a fixed flow rate, you can increase the pipe size. If your design has a fixed pipe size, you can reduce the flow rate to determine what flow it can handle and from this you can determine the accompanying pressures and temperatures.

REFERENCES

1. Street, R. L., G. Z. Watters, and J. K. Vennard, *Elementary Fluid Mechanics*, John Wiley & Sons.
2. Shapiro, A. H., *The Dynamics and Thermodynamics of Compressible Flow*, Vol. 1, John Wiley & Sons, 1953.